Differential-Turning Tactics

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Differential-turning tactics are approached employing the formulation and vehicle 'energy' modeling of a companion paper. First, open-loop tactics are examined, then tactics obtained with a system model featuring further order reductions. These suggest candidate closed-loop tactics for the energy-model case. In the course of further study, an investigation of 'tandem-motion' subarcs is carried out. A necessary condition for capture in a turning chase of extended duration is obtained and, with this as a frame of reference, a qualitative description of the candidate closed-loop tactics presented.

Introduction

TURNING chases in air-to-air combat are studied in the following using the differential-game formulation of a companion paper.[1] Energy-state vehicle models are employed and horizontal-plane kinematics are treated in terms of a turn-angle difference representation.

Of the three simplest evasive tactics, out-climbing, out-turning, and out-dashing one's adversary, the first two are encompassed by the modeling adopted. The tactics exploration will focus mainly upon open-loop and closed-loop control policies; however, a necessary condition for capture in a long-duration chase will be obtained which appears useful for design purposes when used in conjunction with the conditions previously presented.[1]

Modeling

The energy-turn modeling and derived capture requirement of Ref. 1 will be reviewed briefly. The equations of motion are

$$\dot{h} = V \sin \gamma \tag{1}$$

$$\dot{\gamma} = \frac{g}{V}\left[\frac{L \cos \mu}{W} - \cos \gamma\right] \tag{2}$$

$$\dot{E} = [(T-D)\,V/W] \tag{3}$$

$$\dot{\chi} = (gL \sin \mu)/(VW) \tag{4}$$

These contain a thrust-along-the-path simplification which, however, is not essential. Here $E = h + V^2/2g$ is specific energy, χ is heading angle, μ bank angle.

When an order reduction to an energy model is effected, the "lost" boundary conditions are the two on altitude and path angle, this shortcoming of the reduced-order approximation to be alleviated with a "boundary-layer" correction.[2,3] The order-reduction yields the relationships $\sin \gamma = 0$ and

$$L \cos \mu = W \cos \gamma \tag{5}$$

These determine path angle γ and lift coefficient C_L.

$$\gamma = 0 \tag{6}$$

$$C_L = (W/qS \cos \mu) \tag{7}$$

The inequality constraint on lift coefficient

$$C_L \leq \bar{C}_L(M) \tag{8}$$

translates into

$$(W/qS \cos \mu) \leq \bar{C}_L(M) \tag{9}$$

which, for purposes of the reduced-order solution, is an inequality constraint involving both state and control variables. The highest altitude allowed by this constraint, which occurs for $\mu = 0$, must, at the terminal point of the path, equal or exceed the specified final altitude. Evaluated at the specified altitude, the constraint thus becomes a terminal state inequality restricting the final specific cenergy. At each energy level, there is an upper bound on altitude imposed by vertical equilibrium and maximum lift coefficient

$$h_L(E) - h \geq 0 \tag{10}$$

termed the aircraft's "loft-ceiling".

Capture Requirements and Altitude-Dynamics Idealizations

In the energy-modeled turning chase formulation of Ref. 1, capture is defined in terms of angular closure (heading-match) with the pursuer's loft-ceiling equal to or exceeding the evader's. In the case of angular closure with the loft-ceiling-match requirement not met, the question of instantaneous altitude-match must receive special attention since the altitudes of both energy-modeled vehicles are control variables, hence can jump instantaneously. This is dealt with by considering, like Marchal,[4] two extremes of idealization and an intermediate one. One extreme of altitude-dynamics treatment is the assumption that the pursuer is faster than the evader in the sense that the sum of his perception and actuation delay times is small compared to the evader's actuation delay; this results in a pursuer with free choice of altitude between declared limits defining some band within his attainable range of altitudes from his loft-ceiling downward to whatever lower bound may exist, but an evader obliged to remain outside the pursuer's range of altitude choice. The other, "fast evader", extreme leads to free choice of altitude within their respective bounds for both participants. The intermediate idealization is that the pursuer may drive the evader upward to the pursuer's declared limit, but must remain at that altitude himself if he chooses to do this; otherwise, both have free choice of altitude within their respective bounds. This intermediate or neutral case (Marchal's term for a different but somewhat similar case) is really one of a pursuer comparatively fast "in the small", i.e., when near the evader. It appears to be the idealization of main interest. It should be noted that noninfinitesimal altitude capture-tolerance might be introduced in lieu of the match

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requirement to account approximately for the capabilities of tail-aspect missile weaponry.

Open-Loop Optimal Tactics

Consider a version of the pursuit/evasion problem simplified to the selection of energy-turn trajectory pairs for the two craft. The characterization can be made in terms of three parameters in addition to the specified initial energies and the angular separation: the two terminal specific energy values and the elapsed time to capture. The elapsed time could be thought of as determined by the initial angular separation, given the energy final values, and these parameterize the family of trajectory pairs. A candidate is the energy-open final value for the evader paired with the pursuer's energy-open value, if this satisfies the terminal loft-ceiling inequality, otherwise with the threshold pursuer energy for loft-ceiling match.

Energy-turn maneuvers of extended duration for a single craft have the general character of an initial transient, a circling at maximum sustainable turn rate, and a final transient, as discussed in earlier publications; in fact, the two transients can be characterized as boundary layers in a further-reduced problem.[2,3] If the final energy for an extended-duration maneuver is unspecified, the last phase of the final transient is flown at maximum lift to convert energy remaining to heading change, and the open final energy value will be determined as that for which the maximum-lift, zero-altitude turn rate equals the maximum sustainable turn rate.

Because the angles χ_i do not appear in the right members of Eqs. (1-4), the boundary-layer differential equations of the further-reduced model are identical to the Euler equations for the energy model, so that the boundary-layer solutions can be generated from an energy-turn computer program such as that described in Ref. 5. Three boundary-layer transitions from and to the maximum sustainable turn rate condition are shown in Figs. 1-3 for an aircraft of type A (previously described[1,2,6]).

Estimates may be provided for extended-duration cases by further time-scale separation.[2,3] The heading changes are given approximately by

$$\chi_{fi} - \chi_{oi} = \Delta\chi_{oi} + \dot{\chi}_i (t_f - t_o) + \Delta\chi_{fi} \tag{11}$$

where $i=1$ for the evader and $i=2$ for the pursuer. The $\dot{\chi}_i$ are the maximum sustainable turn rates for the two aircraft and the $\Delta\chi_i$ are corrections derived from "boundary-layer" transitions between the energy level of the steady turn and the endpoint energy values by integrating the deviations from the steady value over the initial and terminal boundary layers. Each $\Delta\chi$ value is a function of the corresponding endpoint energy.

The duration of the solution $t_f - t_o$ can be determined explicitly in this approximation from equality of the final headings χ_{1f} and χ_{2f}. This is computed for the evader terminal energy and the pursuer terminal energy taken as the open or loft-ceiling-match threshold value, as just discussed.

The family of energy-turn trajectories for each vehicle has the property of simple covering, hence each member is open-loop optimal. The pairing of such trajectories generates tactics which implicitly assume open-loop optimality, hence, are suspect. However, since differential-turn trajectories can reasonably be expected to possess the open-loop optimality property in the vicinity of the terminal capture point, they are of at least limited interest.

In the simple pairing concept, three parameters are chosen subject to two constraints. The evader's terminal energy choice can be viewed as the independent parameter to be chosen for a maximum of the time of capture. This simplistic view may encounter various difficulties, an obvious one being the sometimes possible choice by the evader of an alternative to the open-end-value for his energy, such as an energy so high that the pursuer cannot manage a loft-ceiling match.

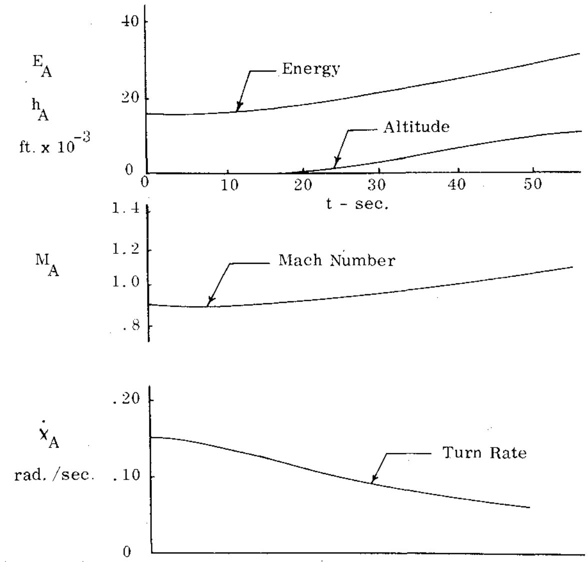

Fig. 1 Transition from maximum sustainable turn to higher E—aircraft A.

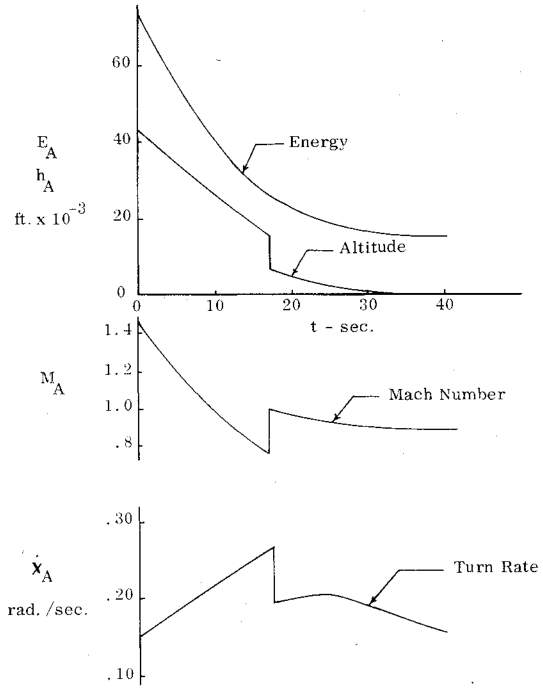

Fig. 2 Transition to maximum sustainable turn from higher E—aircraft A.

Another is that apparently successful pairing may result in the pursuer crossing in front of the evader, i.e., overtaking angularly at intermediate points along the solution generated, which result represents conduct unbecoming an evader, viz., failing to reverse his direction of turn in the face of a negative angular gap. These considerations encourage investigation beyond simple pairing, particularly in problems for which a pursuer sustainable-turn-rate superiority does not exist at high energies with the loft-ceiling inequality satisfied, for here there are bound to be some successful and nearly successful upward-spiralling evasions in the family and tactics different from the open-loop variety.

Tactics with Models Further Reduced in Order

Another form of minimax-time-optimal differential-turning tactics may be discerned from a look at the corresponding problem for vehicle models further reduced in or-

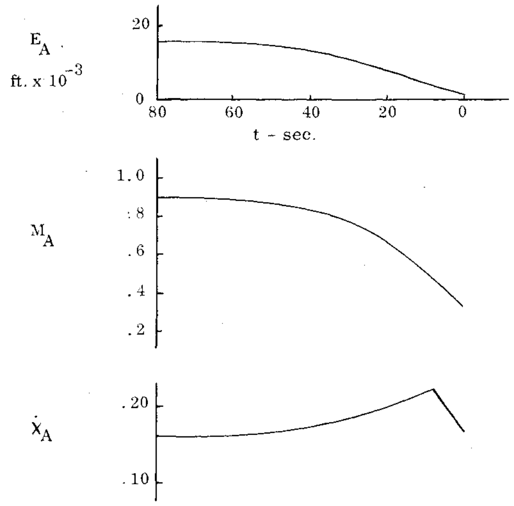

Fig. 3 Transition to maximum sustainable turn from lower *E*—aircraft A.

der. Modeling of this type was employed in the preceding section primarily as a means of representing families of individual aircraft energy-turn trajectories. The further reduction leads to the instantaneously-variable-speed (and energy) models of Refs. 2 and 3, with boundary-layer fairings to approximate energy transitions at the endpoints and at any internal points exhibiting energy jumps.

Along such reduced solutions, the right members of Eqs. (1-3) vanish, h, γ, and E, as well as μ, C_L, and throttle η, are control-like, and the heading difference $\Delta\chi$ is the only state-like variable. Minimax of $\Delta\dot{\chi}$ subject to the constraints produces turning of each vehicle at its highest sustainable rate except at the final point where the minimax is subject to loft-ceiling and power-ceiling inequalities. Capture depends upon a nonnegative margin of pursuer over evader sustainable turn rate, with pursuer energy satisfying both inequalities over the whole spectrum of evader energy.

The Vasil'eva composite, including boundary-layer corrections, then consists of transitions from the energies specified initially for the two vehicles to those values at which they develop their highest steady turning rates, the chase then continuing at these constant but generally unequal turn rates until closure is approached, when another transition takes place to the energy values for the smallest steady turn-rate advantage of pursuer over evader subject to the inequalities.

Energy-Model Tactics Possibilities

There is a temptation to conjecture that the solution of the energy-modeled problem version generally apes that of the super-reduced model, differing mainly in higher-fidelity maneuver details; however, two counterexamples fall readily to hand. One is capture during the initial transient of an evader having superior sustainable-turn-rate capability by a pursuer having an initial energy advantage. The over-simplified model predicts no capture at all in such a situation, this being simply a case of the unsuitability of a singular perturbation approach. A second example is provided by terminal closure occurring with a margin of sustainable turn rate, but with the maximum instantaneous turn rate of the pursuer less than that of the evader, so that capture is, in fact, not possible in the energy-modeled problem version at the energies given by the solution with the simpler model. One suspects that, in at least some such cases, the energy-model solution resembles that given by the approximation with an additional phase, consisting of a turning chase down to low energies, with capture then occurring. The tactics during this final phase can be expected to be open-loop optimal, as described in an earlier section. In the next section, another piece of the jigsaw puzzle will be investigated: subarcs of "tandem motion", along which the turn-angle difference $\Delta\chi$ is zero, but the loft-ceiling match requirement is not met, as at the juncture of the two phases. Then a conjectured closed-loop composite tactic will be described qualitatively.

Tandem Motion

Motion with the pursuer's heading angle χ equal to the evader's, but with the loft-ceiling inequality requirement for capture not met, may be termed tandem motion. A study of such motion may proceed by imposing the heading equality as a hard constraint for the purpose of generating candidates, then screening such candidates using the Euler equations for the differential game (and, in principle at least, other necessary conditions). From such an investigation emerge candidates which, with one exception, have time-variable multipliers λ_χ that fail to satisfy the Euler equation $\dot{\lambda}_\chi = 0$. The exception is symmetric flight, $\dot{\chi} = 0$. Attention will accordingly be confined to this case in the following.

With angular closure effected, the pursuer has two options, in the case of neutral altitude dynamics, and chooses between them so as to minimize the Hamiltonian. One is to follow angularly to maintain closure without threatening altitude-match; in this case, both participants adjust their altitude controls for maximum energy rate, each without regard to the other's choice. A second pursuer option is to press the attack, driving the evader outside a pursuer-chosen range of attainable altitudes. With the neutral modeling, the pursuer must maintain his altitude control against one of the bounds of this chosen range to do this; thus both craft may fly along at the pursuer's loft-ceiling altitude.

If the evader's maximum instantaneous turn rate exceeds the pursuer's, the evader has the option of opening the angular gap at least momentarily (i.e., separating), and may do so either immediately on closure or after a period of tandem motion. A situation calling for tandem motion with deferred separation will be mentioned in the next section.

The two types of symmetric tandem motion have a welcome simplicity. The first, done with unmatched altitudes, consists of minimum-time climbs to higher energy done independently by the two combatants; the trajectories are members of the energy-turn families needed in other phases discussed earlier. The second, carried out with altitudes matched at the pursuer's loft-ceiling value, has all controls determined and can be calculated once and for all by numerical integration of the energy-rate differential equations, the specific energy values at initiation of the tandem symmetric motion being parameters of the family. Tandem motions with $\Delta\chi = 0$ lie in the state subspace of the two specific-energy variables. The regions of tandem motion in the subspace are shown in Fig. 4 for the case of aircraft B as pursuer and A as evader. One boundary of the region, the lower right one, is the separatrix dividing those matched-altitude trajectories that eventually cross into the target set from those that do not. It is of interest to compare time-rates-of-change of loft-ceilings for the two aircraft in 2-D flight at their loft-ceiling altitudes with the energies chosen for equal loft-ceilings. Such a comparison is given for the data of the aircraft A and B in Fig. 5. 2-D tandem motion at the pursuer's loft-ceiling can be ruled out a priori if the evader's curve lies completely above the pursuer's. Thus with A chasing B, no high 2-D tandem trajectories appear, i.e., A in the role of pursuer could not contrive to match loft-ceilings, starting from a near match, by driving his opponent aloft.

Chattering Singular Arc

Motion characterized by chattering between the two types of tandem motion is a possibility. Under a mild simplifying assumption, this appears as a singular arc of a "relaxed" optimal control problem. The assumption is that the energy-rate of the evading craft as a function of altitude at fixed specific

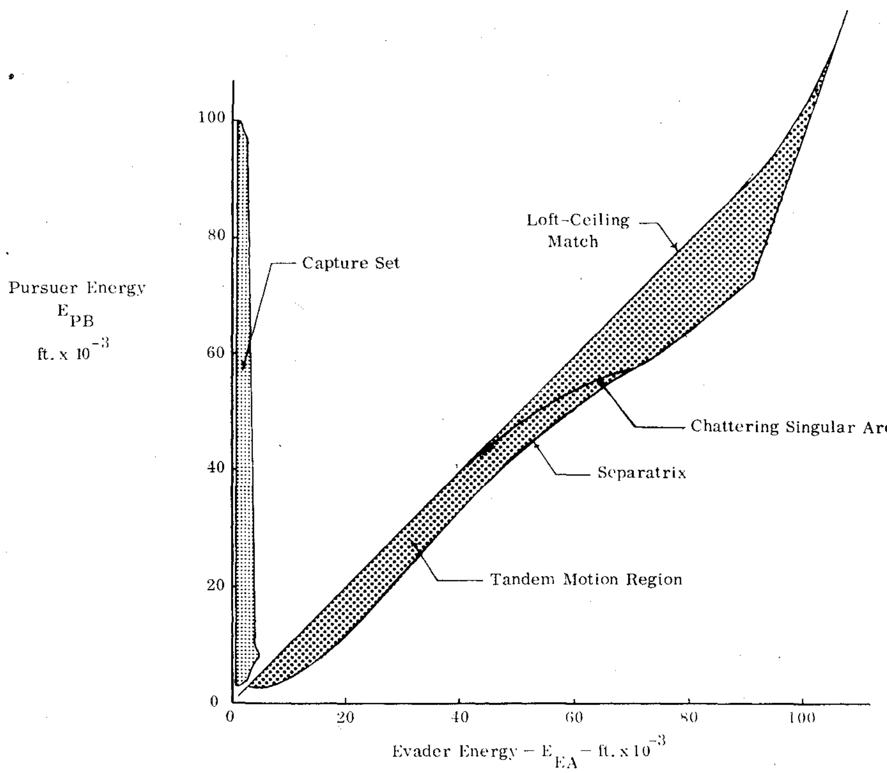

Fig. 4 Tandem-motion regions in energy space.

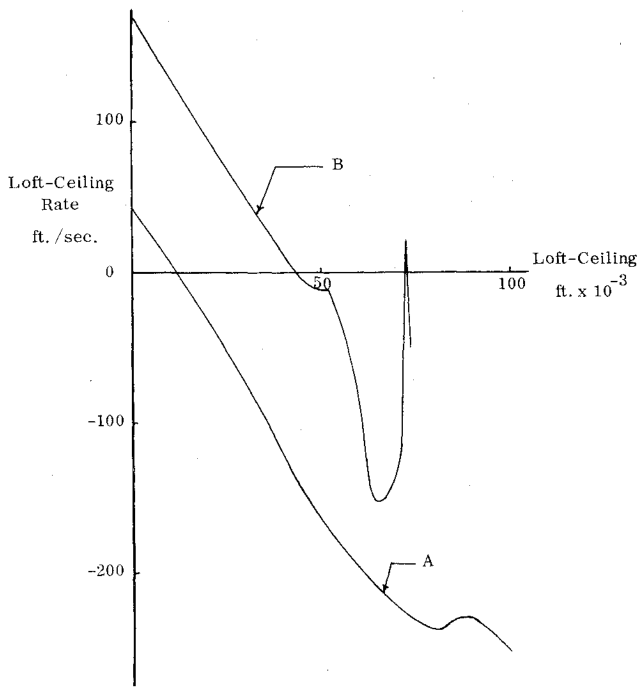

Fig. 5 Loft-ceiling rate at loft-ceiling altitude vs loft-ceiling.

energy has only one peak; this insures that it is optimal for the evader to stay "just above" the pursuer in altitude, or, unless a tolerance is specified, to match the pursuer's altitude.

The problem is reformulated in "relaxed" form, with motion assumed two-dimensional and at full throttle for both players. The assumptions that the evader follows in altitude during the high tandem motion and that both pursuer and evader revert to their individual altitudes for maximum energy rates during the other type makes the problem an optimal control problem. The control variable is σ, a parameter of interpolation between "tandem/loft" and "tandem/trail", as the two types of motion might be termed, $0 \leq \sigma \leq 1$. The "relaxed" equations of state are

$$\dot{E}_1 = f_1(E_1, h_1^*) + \sigma[f_1(E_1, h_{2L}) - f_1(E_1, h_1^*)] \tag{12}$$

$$\dot{E}_2 = f_2(E_2, h_2^*) + \sigma[f_2(E_2, h_{2L}) - f_2(E_2, h_2^*)] \tag{13}$$

where

$$f_1^* \equiv \left[\frac{V_1(T_1 - D_1)}{W_1} \right]_{h_1^*} \tag{14}$$

$$f_2^* \equiv \left[\frac{V_2(T_2 - D_2)}{W_2} \right]_{h_2^*} \tag{15}$$

$$f_{2L} \equiv \left[\frac{V_2(T_2 - D_2)}{W_2} \right]_{h_{2L}} \tag{16}$$

$$f_{12L} \equiv \left[\frac{V_1(T_1 - D_1)}{W_1} \right]_{h_{2L}} \tag{17}$$

In these expressions, $h_i^*(E_i)$ is the altitude for maximum energy rate f_i^*, $i=1$ for the evader and $i=2$ for the pursuer. The pursuer's loft-ceiling altitude is designated h_{2L}; it is a function of the pursuer's specific energy E_2.

A singular arc of first-order variety[8] is found in energy space by forming a Hamiltonian $\mathcal{H}$ for the relaxed problem in the usual way and observing that the two conditions

$$\frac{\partial \mathcal{H}}{\partial \sigma} = 0 \tag{18}$$

$$\frac{d}{dt} \frac{\partial \mathcal{H}}{\partial \sigma} = 0 \tag{19}$$

comprise a homogenous linear simultaneous algebraic system in the multiplier variables whose determinant Δ must vanish for compatibility

$$\Delta = (f_{12L} - f_2^*)\left[f_1 \frac{\partial f_{12L}}{\partial E_1} - f_{12L}\frac{\partial f_1^*}{\partial E_1} + f_2^* \frac{\partial f_{12L}}{\partial E_2}\right]$$

$$- (f_{12L} - f_1^*)\left[f_2^* \frac{\partial f_{2L}}{\partial E_2} - f_{2L}\frac{\partial f_2^*}{\partial E_2}\right] = 0 \qquad (20)$$

where

$$\frac{\partial f_1^*}{\partial E_1} = \frac{\partial}{\partial E_1}\left[\frac{V_1(T_1 - D_1)}{W_1}\right]_{h_1^*}$$

$$+ \frac{\partial}{\partial h_1}\left[\frac{V_1(T_1 - D_1)}{W_1}\right]_{h_1^*}\frac{dh_1^*}{dE_1} \qquad (21)$$

$$\frac{\partial f_2^*}{\partial E_2} = \frac{\partial}{\partial E_2}\left[\frac{V_2(T_2 - D_2)}{W_2}\right]_{h_2^*}$$

$$+ \frac{\partial}{\partial h_2}\left[\frac{V_2(T_2 - D_2)}{W_2}\right]_{h_2^*}\frac{dh_2^*}{dE_2} \qquad (22)$$

$$\frac{\partial f_{12L}}{\partial E_1} = \frac{\partial}{\partial E_1}\left[\frac{V_1(T_1 - D_1)}{W_1}\right]_{h_{2L}} \qquad (23$$

$$\frac{\partial f_{12L}}{\partial E_2} = \frac{\partial}{\partial h_1}\left[\frac{V_1(T_1 - D_1)}{W_1}\right]_{h_{2L}}\frac{dh_{2L}}{dE_2} \qquad (24)$$

$$\frac{\partial f_{2L}}{\partial E_2} = \frac{\partial}{\partial E_2}\left[\frac{V_2(T_2 - D_2)}{W_2}\right]_{h_{2L}}$$

$$+ \frac{\partial}{\partial h_2}\left[\frac{V_2(T_2 - D_2)}{W_2}\right]_{h_{2L}}\frac{\partial h_{2L}}{\partial E_2} \qquad (25)$$

As the vehicle energies decrease in the course of tandem motion at the pursuer's loft-ceiling, a transition to tandem/trail may occur; but then the trajectories gain energy and may undergo a transition to tandem/loft motion. The limit of a sequence of such transitions is a chattering motion described by the singular arc of the relaxed problem. If the chattering motion reaches the loft-ceiling-match curve, a transition to a high turn-rate open-loop trajectory pair occurs, assuming the pursuer's maximum instantaneous turn rate is exceeded by the evader's; otherwise it may occur earlier. (In the example, the evader happened to have higher instantaneous turn rate at matched loft-ceilings.)

The chattering singular arc determined computationally from Eq. (20) for the case of B pursuing A is shown in Fig. 4.

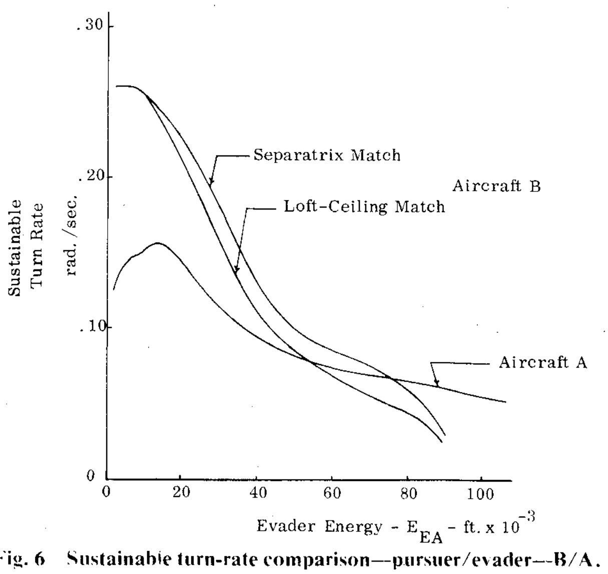

Fig. 6 Sustainable turn-rate comparison—pursuer/evader—B/A.

Testing indicates that no segment of the arc shown meets simultaneously the necessary condition for optimality of Ref. 8 and the requirement $0 \leq \sigma \leq 1$. It thus appears that the chattering singular motion does not play a role in the particular example; however, there is no reason to expect this to be generally true.

Necessary Condition for Capture

Regard two aircraft circling at constant but generally unequal turn rates, and consider how the one designated as evader might choose his own specific energy level. Plot his maximum sustainable turn rate at each energy level vs energy level. Now for computation of pursuer comparison turn rate, require that the pursuer choose his specific energy level high enough that it equals or exceeds the separatrix value, if there is a separatrix, otherwise high enough that his loft-ceiling equals or exceeds the evader's. Plot the pursuer's turn rate, determined as the highest sustainable within this constraint on his energy, on the same figure vs evader energy. The evader then may be assumed to choose the energy level affording the pursuer the smallest turn-rate advantage. The process just sketched is that of minimaxing the turn-rate difference subject to the inequality constraint. Each point of the pursuer's curve must lie on or above the evader's to permit capture, i.e., angular closure with the constraint met.

Because the maneuvering tactics used to derive the criterion are rather special, viz., constant-rate turns, it does not represent a condition necessary for capture in the differential-game setting. A necessary condition for capture in an important sub-class of solutions may be obtained, however, by retaining constant-turn-rate evasion, allowing the evader choice of his own energy level, and restricting attention to chases with low initial pursuer energy. This restriction screens out early capture attributable to high energy of pursuer in relation to evader initially, narrowing the range of situations encompassed to those for which capture occurs as a result of resources inherent in the design of the pursuing vehicle. Thus the criterion may be characterized as a necessary condition for capture in an extended-duration turning chase starting from any initial state.

Sustainable-turn-rate data for aircraft A and B are given.[1,6] These are combined with the loft-ceiling and separatrix computational results to produce the turn-rate comparison required for the necessary condition for capture in an extended-duration chase in Fig. 6. If B is designated pursuer (Fig. 6), the necessary condition can be stated as a requirement that the curve for aircraft A lie below a derived curve for B which is the composite of a horizontal segment to the left of the peak and the B-curve to the right. The necessary condition is not met, either for B chasing A or A chasing B in a similar comparison. As suggested by a view of the envelope and sustainable turn-rate data (as by overlay in Ref. 6), the outcome of a turning duel will depend strongly upon initial energies and angular separation.

The following argument can be given for the necessary condition for extended-duration capture, which is that the pursuer's maximum sustainable turn-rate equal or exceed the evader's over the attainable range of evader energies, with the pursuer's energy chosen in the comparison to equal or exceed the separatrix value, or the loft-ceiling-match value, if there is no separatrix. Assume the evader turns steadily at maximum sustainable turn rate at the energy for which the pursuer's turn rate margin is the least, negative margins not being excluded. In a long-duration chase, non-negative margin is required to close. If the pursuer presses the attack with energy-deficient closure, i.e., closure with less than either separatrix or loft-ceiling-match energy, then the loft-ceiling gap widens indefinitely. Thus, failure of the pursuer to meet the sustainable turn-rate requirement of the necessary condition results in either no closure or closure without subsequent capture.

The necessary condition, as stated, applies to the neutral model of altitude dynamics. It may be noted that a "fast-evader" model leads to a simpler result: a sustainable-turn-rate superiority requirement at match loft-ceiling. Further investigation of solution families for this model is of interest for future research.

Maneuver Sequences

A composite of the various pieces discussed is suggested in the following qualitative description of maneuvering sequences resulting in capture. It is based upon the conjectured sequence of the preceding sections with some simplifications to be noted.

If there is considerable angular separation, the evader will first tend toward the specific-energy level corresponding to his highest sustainable turn rate, then as the pursuer closes the gap, toward the specific-energy level most favorable to him in the differential sense, as shown by the comparison of sustainable turn rates used in evaluation of the necessary condition. The pursuer will likewise adjust his energy level toward the value at which he develops his best sustainable turn rate when the situation presages a long-duration chase (i.e., when his highest sustainable rate exceeds the evader's), then, as he closes angularly, will bring his energy toward the value for loft-ceiling match. If the pursuer approaches angular closure with at least loft-ceiling-match energy, but the evader's maximum instantaneous turn rate exceeds that of the pursuer, a transition occurs to a high turn-rate spiral for both aircraft, i.e., open-loop optimal turning flight, proceeding to low energy levels, where capture occurs.

If the pursuer closes angularly and presses the attack with insufficient energy to match loft-ceilings, the evader ascends. Both craft then fly along in 2-D at essentially the pursuer's loft-ceiling altitude. Both pursuer and evader may lose energy during this phase, as the pursuer's loft-ceiling altitude may be outside both flight envelopes. The 2-D chase may sink to lower altitude as the pursuer's loft-ceiling altitude, like his energy level, drops off. This pursuit tactic of deliberately closing with an energy deficiency, hence forcing the chase upward, is a hopeful one provided the pursuer's energy is at least equal to the separatrix value, for the loft-ceiling gap will then subsequently close while the chase remains 2-D. Since the asymmetric tandem motion has been ruled out, the chase can go 3-D only if the evader has superior maximum turn rate, which would make separation possible.

If the aircraft data are such that the necessary condition is not met, i.e., there exists a range of evader energy over which the pursuer's sustainable turn rate margin is in fact negative, then successful evasion may be possible depending on the initial separation and energies. The evader will generally, in this case, adjust his energy toward this favorable range immediately, at sacrifice of turn rate, unless the angular gap has become small, foregoing the usual intermediate adjustment toward his maximum-sustainable-turn-rate energy range. The details of successful evasion trajectories depend upon an auxiliary performance index, such as closest-approach angular gap, to be minimaxed, as minimax time-to-capture obviously does not apply.

The discussion of maneuver sequences has been oversimplified in several respects, one of the most obvious being that a closed-loop representation of control policies must be employed for an accurate description, and no effort has been made at this. Another is that it conveys the false impression that approach to closure with the pursuer slightly exceeding the requirements of the necessary condition will result in eventual capture, i.e., that a strengthened form of the condition is sufficient. Such a guarantee cannot be issued without additional requirements. A rather conservative combination of conditions providing a guarantee is given in Ref. 1. Pursuer superiority in maximum instantanous turn-rate, maximum sustainable turn-rate, and maximum energy-rate, all simultaneously at energies for matched loft-ceiling, over the entire energy range, as implied by envelopment of the evader's hodograph figure by the pursuer's. Some less conservative combination yet to be found may also serve; however, even the least restrictive one probably must limit in some way simultaneous evader superiority in maximum instantaneous turn-rate and in maximum energy rate. A third oversimplification is the neglect of transition possibilities from the high 2-D tandem motion and the rules governing them, which are being investigated.[3] A fourth is the omission of chattering between the two types of tandem motion or a flyable approximation to it.

Generalized Corner Condition

The indirect method of the calculus of variations, which is based mainly upon the generation of candidates from first-order necessary conditions, encounters an obstacle in differential game problems in that the Erdman-Weierstrass condition, assuring continuity of the multiplier variables, does not apply. Indeed the multiplier variables may jump along various types of singular surface in the state space.In the present problem, the vanishing of the heading-difference state variable defines such a surface, a switching surface, when the evader's craft has superiority in maximum turn rate.

The generalized corner condition requires that any jump in the multiplier vector be normal to the singular surface. In the case just noted, the condition thus implies that only the heading multiplier component may jump. The magnitude of the jump in the multiplier is not determined by local properties. A singular subarc instead of a simple switch may occur: 2-D tandem motion following angular closure. In this case, the heading multiplier evidently jumps to zero upon entry to the singular surface; however, the time of exit (if any) and the corresponding multiplier jump are not determined locally.

Conclusions

The present paper has discussed differential-turning trajectory pairs, their composition and, qualitatively, the implications for minimax-optimal tactics. A necessary condition for capture occupies a central position in the theory and, together with the sufficient condition presented in the companion paper,[1] should prove useful in design studies of maneuvering performance. It also points the way toward computation of trajectory pairs of particular interest, those which define the boundary surface in state space separating successful pursuit from successful evasion.[9]

References

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